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METHODS OF CONSTRUCTION OF PROPER BALANCED TERNARY DESIGNS

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SUMMARY

Construction of balanced ternary designs has been resorted to based on BIB designs which are introduced in theorems 2.1 and 2.2. In these designs the number of blocks as well as the size of the blocks is much less as compared to the designs due to John [3] and Das and Rao [1]. Theorem 2.3 gives the construction of balanced ternary designs from α-resolvable BIB designs which is an improvement over the result due to Dey [2]. Further, the result due to Tyagi and Rizwi [9] can be deduced from the results obtained.

Keywords: Projective geometry of two dimensions; Kroneckor product; Resolvable BIB design.

Introduction

Eversince balanced n-ary designs were introduced by Tocher [8] seldom have basic methods been suggested for the construction of these designs. Tocher himself obtained the designs, which were all proper, by trial and error. John [3], Das and Rao [1], Kulshrestha [4], Nigam [6], Nigam *et al.* [7], Tyagi and Rizwi [9] all started their construction with the existing BIB designs. Murthy and Das [5] made a distinct approach which in a way is based on orthogonal arrays. Das and Rao [1] resorted to the construction of n-ary designs by taking the product of appropriate BIB designs. Dey's [2] interest centred round the construction of balanced n-ary designs from affine α-resolvable BIB designs. In this paper we have employed in the first place Galois field for the construction of balanced ternary designs. Secondly the conventional approach based on BIB designs has been resorted to in some cases. Further most of the traditional methods have been shown to be desirable by Kroneckor product.

2. Methods of Construction

Construction of balanced ternary designs using Finite Geometrices is contained in the following theorems.

THEOREM 2.1. *Existence of EG(2, s), Eucledian geometry of two dimensions, implies the existence of a ternary balanced proper design with parameters* $v = s^2$, $b = s^2(s + 1)$, $r = (s + 1)^2$, $k = s + 1$, $\lambda = s + 2$.

Proof. $EG(2, s)$ can be obtained from $PG(2, s)$, Projective geometry of two dimensions, by removing the line at infinity and the points thereof.

Suppose out of a line we form s lines by taking all the points on the line with one distinct point occuring twice in each line. Then we will have $s^2(s + 1)$ lines each containing $s + 1$ points. Since $s + 1$ lines pass through a point and s lines are formed out of a line each point, will be repeated $(s + 1)^2$ times in these lines. One line can pass through two distinct points and in the s lines formed out of this line this pair occur $s + 2$ times so that $\lambda = s + 2$. Identifying the lines as blocks and the points as treatments we get a ternary design with parameters indicated as above.

Example 2.1

The parallel pencils of $EG(2, 3)$ are

1	2	3	1	4	7	1	6	8	1	5	9
4	5	6	2	5	8	4	2	9	3	8	4
7	8	9	3	6	9	3	5	7	2	6	9

Blocks are formed by repeating a distinct point or a line and they are

(1 2 3 3)	(1 4 7 7)	(1 6 8 8)	(1 5 9 9)
(1 2 2 3)	(1 4 4 7)	(1 6 6 8)	(1 5 5 9)
(1 1 2 3)	(1 1 4 7)	(1 1 6 8)	(1 1 5 9)
(4 5 6 6)	(2 5 8 8)	(4 2 9 9)	(3 8 4 4)
(4 5 5 6)	(2 5 5 8)	(4 2 2 9)	(3 8 8 4)
(4 4 5 6)	(2 2 5 8)	(4 2 2 9)	(3 3 8 4)
(7 8 9 9)	(3 6 9 9)	(3 5 7 7)	(2 6 7 7)
(7 8 8 9)	(3 6 6 9)	(3 5 5 7)	(2 6 6 7)
(7 7 8 9)	(3 3 6 9)	(3 3 5 7)	(2 2 6 7)

which form the blocks of a balanced ternary design with parameters

$$v = 9, \quad b = 36, \quad r = 16, \quad k = 4, \quad \lambda = 5.$$

THEOREM 2.2. *Existence of $PG(2, s)$ implies the existence of balanced proper ternary design with parameters* $v = s^2 + s + 1, b = (s + 1)(s^2 + s + 1), r = (s + 1)(s + 2), k = s + 2, \lambda = (s + 3)$. Proof of the theorem is in the same lines as that of theorem 2.1.

THEOREM 2.3. *Let M be a module consisting of the elements $\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n-1)}$. Attatch to each element m varieties. The varieties $\alpha_j^{(0)}, \alpha_j^{(1)}, \ldots, \alpha_j^{(n-1)}$ are said to belong to the jth class. Form the blocks $B_1, B_2, \ldots, B_t$ such that*

(1) *Every block consists of k varieties of which $k - 1$ alone are distinct;*
(2) *The differences except zero differences arising from the varieties are symmetrically repeated each occuring λ times;*
(3) *The zero differences are all pure and arise equally frequently from all classes;*
(4) *Among the elements of the blocks exactly r varieties belong to each of the m classes.*

If θ is an element of the module form blocks $B(i, \theta)$ by adding to the elements of the ith block $i = 1, 2, \ldots, t$, $\theta = \alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(n-1)}$. The blocks obtained will form a ternary design with parameters $v = nm$, $b = nt$, r, k, λ.

Proof. The expressions for v, b and k are evident. Since a variety occurs twice in the initial blocks the design is ternary.

Every mixed difference and every pure nonzero difference occurs λ times in the initial blocks thereby showing that the number of pairs a variety makes with another variety is the same, namely λ. Therefore the final blocks will be such that the number of pairs a variety makes with another variety is λ.

Since every zero difference is pure and occurs equally frequently in the initial blocks such that they are equally repeated in every class every treatment will occur r times in the final blocks.

Example 2.2

Consider the module $M = (0, 1, 2, 3) \bmod 4$ and attach two varieties to each element of the module. Take the initial blocks as

$$\begin{array}{ll} (0_1 \;\; 0_2 \;\; 2_1 \;\; 2_1) & (0_2 \;\; 2_1 \;\; 3_2 \;\; 3_2) \\ (0_1 \;\; 0_2 \;\; 0_2 \;\; 2_1) & (0_2 \;\; 2_1 \;\; 2_1 \;\; 3_2) \\ (0_1 \;\; 0_1 \;\; 0_2 \;\; 2_1) & (0_2 \;\; 0_2 \;\; 2_1 \;\; 3_2) \end{array}$$

Adding each element of the module to these blocks we have the blocks

$(0_1\ 0_2\ 2_1\ 2_1)$	$(1_1\ 1_2\ 3_1\ 3_1)$	$(2_1\ 2_2\ 0_1\ 0_1)$	$(3_1\ 3_2\ 1_1\ 1_1)$
$(0_1\ 0_2\ 0_2\ 2_1)$	$(1_1\ 1_2\ 1_2\ 3_1)$	$(2_1\ 2_2\ 2_2\ 0_1)$	$(3_1\ 3_2\ 3_2\ 1_1)$
$(0_1\ 0_1\ 0_2\ 2_1)$	$(1_1\ 1_1\ 1_2\ 3_1)$	$(2_1\ 2_1\ 2_2\ 0_1)$	$(3_1\ 3_1\ 3_2\ 1_1)$
$(0_2\ 2_1\ 3_2\ 3_2)$	$(1_2\ 3_1\ 0_2\ 0_2)$	$(2_2\ 0_1\ 1_2\ 1_2)$	$(3_2\ 1_1\ 2_2\ 2_2)$
$(0_2\ 2_1\ 2_1\ 3_2)$	$(1_2\ 3_1\ 3_1\ 0_2)$	$(2_2\ 0_1\ 0_1\ 1_2)$	$(3_2\ 1_1\ 1_1\ 2_2)$
$(0_2\ 0_2\ 2_1\ 3_2)$	$(1_2\ 1_2\ 3_1\ 0_2)$	$(2_2\ 2_2\ 0_1\ 1_2)$	$(3_2\ 3_2\ 1_1\ 2_2)$

and these blocks form the blocks of a balanced ternary design with parameters $v = 8$, $b = 24$, $k = 4$, $r = 12$, $\lambda = 10$.

THEOREM 2.4. *Let there be a BIB design with parameters v, b, r, k, λ. From each block of the BIBD construct k blocks each of size $k + 1$ with block content as all treatments of the blocks with one distinct treatment repeated in a block. The resulting design will be a ternary design with parameters $v' = v$, $b' = kb$, $r' = r(k + 1)$, $k' = k + 1$, $\lambda' = \lambda(k + 2)$.*

The theorem indicates construction of ternary designs from existing BIB designs.

Proof. The expressions for b', v' and k' are obvious and the design is ternary.

In the original design a treatment occurs in r blocks and from each block of the original design k blocks are formed in which a treatment of the original block will occur $k + 1$ times. Hence the number of replications of a treatment in the final blocks will be $r(k + 1)$.

Since two treatments occur together in blocks from each of which k blocks are formed by repeating a distinct treatment once in each of the new blocks the number of pairs a treatment makes with another treatment is $\lambda(k + 2)$.

THEOREM 2.5. *Let there be an α-resolvable BIB design with parameters v, b, r, k, λ. If we collapse distinct blocks of this design in pairs omitting those combinations arising from the blocks within each α-replicate the design obtained will be a balanced ternary design having parameters $v' = v$, $b' = \frac{1}{2} n^2 t(t - 1)$, $k' = 2k$, $r' = n^2 t(t - 1)\, k/v$, $\lambda' = n(t - 1)\, \lambda + r(r - \alpha)$.*

The theorem embodies the construction of ternary designs from resolvable BIB designs.

Proof. Let N_1 be a BIBD with parameters v, b, r, k, λ. Then collapsing each of its blocks with every block of the design is equivalent to taking the sum $E(1, b) X N_1' + N_1 X E(1, b) = N$, say where X indicates Kroneckor product. Then,

$$NN' = 2b\, N_1N_1' + 2r^2\, E(v, v)$$

This matrix has got all diagonal elements equal and all off-diagonal elements equal. Further the block size is $2k$. Hence the design N is balanced ternary.

Let $N_1 = (M_1\, M_2 \ldots M_t)$ be an resolvable BIB design where M_i is a $v \times n$ matrix consisting of a single replicate. Then collapsing the blocks of M_i is equivalent to taking the sum

$$E(1, n) \times M_i + M_i \times E(1, n) = J_i$$

$$J_iJ_i' = 2n\, M_i M_i' + 2\alpha^2\, E(v, v)$$

$$\sum_1^t J_iJ_i' = 2n\, N_1\, N_1' + 2t\, \alpha^2\, E(v, v)$$

If P denotes the incidence matrix obtained by collapsing blocks of N_1 excepting those generated from within each of $M_1, M_2, \ldots, M_t$.

$PP' = 2b\, N_1\, N_1' + 2r^2\, E(v, v) - 2n\, N_1 N_1' - 2t\, \alpha^2\, E(v, v)$. Now P contains $b^2 - n^2t$ blocks. If r' is the number of replications of a treatment in P, the C-matrix of P is

$$C = r'\, I_v - \frac{1}{2k}\, PP'$$

Therefore,

$$r' = \frac{1}{2k}\, [2(b - n)\, \{r + \lambda\, (v - 1)\} + 2(r^2 - t\alpha^2)\, v]$$

i.e. $r' = 2n^2t\, (t - 1)\, k/v$.

Number of pairs a treatment makes with another treatment is $2(b - n)\, \lambda + 2(r^2 - t\alpha^2) = n(t - 1)\, \lambda + r\, (r - \alpha)$. When we collapse only distinct pairs of blocks the design obtained will have parameters $v' = v$, $r' = n^2t(t - 1)\, k/v$, $k' = 2k$, $\lambda' = n(i - 1)\, \lambda + r(r - \alpha)$ $b' = \frac{1}{2}\, (b^2 - n^2t) = \frac{1}{2}\, n^2t(t - 1)$.

Example 2.3

Consider the resolvable BIB design with parameters $t = 4, n = 3,$

$\alpha = 1, v = 9, b = 12, r = 4, k = 3, \lambda = 1$. The blocks are

(1 2 3)	(1 4 7)	(1 6 8)	(1 5 9)
(4 5 6)	(2 5 8)	(4 2 9)	(3 8 4)
(7 8 9)	(3 6 9)	(3 5 7)	(2 6 7)

Collapsing the blocks as stated in the theorem we get a balanced ternary design with parameters $v = 9, b = 54, r = 36, k = 6, \lambda = 15$.

Corollary 2.1. *Let there be a BIB design having parameters v, b, r, k, λ. If N is the design obtained by collapsing pairs of distinct blocks of the BIB design, N is a balanced proper ternary design have parameters $v' = v$, $b' = b(b-1)/2$, $r' = r(b-1)$, $k' = 2k$, $\lambda' = (b-2)\lambda + r^2$.*

Proof. Let the BIB design be denoted by N_1.

Define

$$N_2 = E(1, b) \times N_1 + N_1 \times E(1, b)$$

$$N_2 N_2' = 2b\ N_1 N_1' + 2r^2 E(v, v)$$

Addition of ith block B_i to itself gives rise to a design whose incidence matrix is $2B_i$.

Now $2B_i \cdot 2B_i' = 4B_i B_i'$

$$\sum_1^b 4B_i B_i' = 4N_1 N_1'$$

Subtracting this from $N_2 N_2'$ we have the design N_3 such that

$$N_3 N_3' = 2b\ N_1 N_1' + 2r^2 E(v, v) - 4N_1 N_1'$$

$$= 2(b-2)\ N_1 N_1' + 2r^2 E(v, v)$$

The parameters of N_3 are $v_3 = v$, $b_3 = b(b-1)$, $k_3 = 2k$, $\lambda_3 = 2(b-2)\lambda + 2r^2$, $r_3 = 1/2k\ [2(b-2)\{r + (v-1)\} + 2r^2 v] = 2r(b-1)$

Hence the parameters of N are

$$v' = v,\ b' = \frac{b(b-1)}{2},\ k' = 2k,\ r' = r(b-1),\ \lambda' = (b-2)\lambda + r^2$$

Corollary 2.2. *Let there be a BIB design N_2 having parameters v, b, r, k, λ. If N is the design obtained by collapsing pairs of blocks (i, j) such that $i \leqslant j$ then N is a balanced ternary design with parameters* $v' = v$, $b' = b(b+1)/2$, $k' = 2k$, $r' = r(b+1)$, $\lambda' = (b+2)\lambda + r^2$.

Proof. In Corollary 2.1 we have seen that if N_1 is the design obtained by collapsing pairs of distinct blocks

$$N_1 N_1' = (b-2) N_2 N_2' + r^2 E(v, v).$$

If we add to N_1 the blocks obtained by collapsing a block to itself we get N of this corollary. Then,

$$NN' = (b-2) N_2 N_2' + r^2 E(v, v) + 4 N_2 N_2'$$

$$= (b+2) N_2 N_2' + r^2 E(v, v)$$

Hence parameters of N are $v' = v$, $b' = b(b+1)/2$, $k' = 2k$, $\lambda' = (b+2)\lambda + r^2$, $r' = 1/2k\,[(b+2)\{r + \lambda(v-1)\} + r^2 v] = r(b+1)$.

Corollary 2.3. *If N is an n-ary proper design, collapsing distinct blocks of N we get a proper balanced* $(2n-1)$*-ary design.*

3. Conclusion

Conventional methods used for the construction of BIB designs using Galois field have been extended for the construction of balanced ternary designs and they are embodied in the theorems 2.1, 2.2 and 2.3. In these designs the number of blocks as well as the size of the block is much less as compared to the designs due to John [3], Das and Rao [1], Kulshreshtha [4]. Nigam [6], Nigam *et al.* [7] etc. Theorem 2.5 gives the construction of balanced ternary designs from α-resolvable BIB design which is an improvement over the result due to Dey [2] which requires the existence of affine α-resolvable designs for construction of ternary designs. The Corollaries show that using Kroneckor product the results due to Nigam [6] can be obtained. Further the result due to Tyagi and Rizwi [9] can be deduced from the results of this paper.

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